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Exact Traveling Wave Solutions for Nonlinear PDEs in Mathematical Physics using the Generalized Kudryashov Method

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Abstract: The generalized Kudryashov method is applied in this article for finding the exact solutions of nonlinear partial differential equations (PDEs) in mathematical physics. Solitons and other solutions are given. To illustrate the validity of this method, we apply it to three nonlinear PDEs, namely, the diffusive predator-prey system, the nonlinear Bogoyavlenskii equations and the nonlinear telegraph equation. These equations are related to signal analysis for transmission and propagation of electrical signals. As a result, many analytical exact solutions of these equations are obtained including symmetrical Fibonacci function solutions and hyperbolic function solutions. Physical explanations for some solutions of the given three nonlinear PDEs are obtained. Comparison our new results with the well-known results are given.

Keywords: Nonlinear PDEs, Generalized Kudryashov method, Symmetrical hyperbolic Fibonacci function, Exact solutions, The diffusive predator-prey system, The nonlinear Bogoyavlenskii equations, The nonlinear telegraph equation.

1 Introduction

Many important phenomena and dynamic processes in physics, mechanics, chemistry and biology can be represented by nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in the soliton theory. The explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. These explicit formulas may provide physical information and help us to understand the mechanism of related physical models. In recent years, many kinds of powerful methods have been presented to find the exact solutions of nonlinear partial differential equations, such as the homogeneous balance method [1], the Hirota's bilinear transformation method [2, 3], the tanh-function method [4, 5], the (G'/G)-expansion method [6 – 8], the exp-function method [9, 10], the

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multiple exp-function method [11 – 13], the symmetry method [14, 15], the modified simple equation method [16 – 18], the Jacobi elliptic function expansion [19], the Bäcklund transform [20, 21], the modified extended Fan sub-equation method [22], the auxiliary equation method [23, 24], the first integral method [25], the generalized Kudryashov method [26 – 30], the soliton ansatz method [31 – 57] and so on.

The objective of this paper is to construct the exact solutions of the diffusive predator-prey system [58, 59], the nonlinear Bogoyavlenskii equations [59] and the nonlinear telegraph equation [58] by using the generalized Kudryashov method [30].

The rest of this article can be organized as follows: In Section 2, we give the description of the generalized Kudryashov method. In Section 3, we use this method to solve the diffusive predator-prey system, the nonlinear Bogoyavlenskii equations, and the nonlinear telegraph equation. In Section 4, physical explanations of some results are presented. In Section 5, some conclusions are given.

2 Description of the Generalized Kudryashov Method

Suppose that a nonlinear PDE has the following from:

$$F(u,u_t,u_x,u_{tt},u_{xt},u_{xx},...)=0, \tag{1}$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u=u(x,t)$ and its partial derivatives, in which the highest order derivatives and highest nonlinear terms are involved. The main steps of the generalized Kudryashov method are described as follows:

Step 1. First of all, we use the wave transformation:

$$u(x,t)=U(\zeta), \quad \zeta=kx\pm\lambda t, \tag{2}$$

where k and λ are arbitrary constants with $k,\lambda\neq 0$, to reduce the equation (1) into the following nonlinear ordinary differential equation (ODE):

$$H(U,U',U'',U''',...)=0, \tag{3}$$

where H is a polynomial in $U(\zeta)$ and its total derivatives $U',U'',U''',...$ such that $U'=\mathrm{d}U/\mathrm{d}\zeta$, $U''=\mathrm{d}^2U/\mathrm{d}\zeta^2$ and so on.

Step 2. We assume that the formal solution of the ODE (3) can be written in the following rational form:

$$U(\zeta)=\frac{\sum_{i=0}^{n}a_iQ^i(\zeta)}{\sum_{j=0}^{m}b_jQ^j(\zeta)}=\frac{A[Q(\zeta)]}{B[Q(\zeta)]}, \tag{4}$$

where:

$$Q = 1/1 \pm a^{\zeta}, \; A[Q(\zeta)] = \sum_{i=0}^{n} a_i Q^i(\zeta), \; \mathrm{d}U/\mathrm{d}\zeta \text{ and } B[Q(\zeta)] = m\sum_{j=0} b_j Q^j(\zeta).$$

The function Q is the solution of the equation

$$Q' = Q(Q-1)\ln(a), \qquad 0 < a \neq 1. \tag{5}$$

Taking into consideration (4), we obtain

$$U'(\zeta) = Q(Q-1)\left[\frac{A'B - AB'}{B^2}\right]\ln(a), \tag{6}$$

$$U''(\zeta) = Q(Q-1)(2Q-1)\left[\frac{A'B - AB'}{B^2}\right]\ln^2(a) +$$

$$+Q^2(Q-1)^2\left[\frac{B(A''B - AB'') - 2A'B'B + 2A(B')^2}{B^3}\right]\ln^2(a), \tag{7}$$

$$U'''(\zeta) = Q^3(Q-1)^3\ln^3(a)\times$$

$$\times\left[\frac{(A'''B - AB''' - 3A''B' - 3A'B'')B + 6B'(A'B' + AB'')}{B^3} - \frac{6A(B')^3}{B^4}\right] +$$

$$+3Q^2(Q-1)^2(2Q-1)\left[\frac{(A''B - AB'' - 2A'B')B + 2A(B')^2}{B^3}\right]\ln^3(a) +$$

$$+Q(Q-1)(6Q^2 - 6Q + 1)\left[\frac{A'B - AB'}{B^2}\right]\ln^3(a), \tag{8}$$

and similar for higher order differentiation terms.

Step 3. Under the terms of the given method, we suppose that the solution of (3) can be written in the following form:

$$U(\zeta) = \frac{a_0 + a_1 Q + a_2 Q^2 + \cdots + a_n Q^n}{b_0 + b_1 Q + b_2 Q^2 + \cdots + b_m Q^m}. \tag{9}$$

To calculate the values m and n in (9) that is the pole order for the general solution of (3), we progress conformably as in the classical Kudryashov method on balancing the highest order nonlinear terms and the highest order derivatives of $U(\zeta)$ in (3) and we can determine a formula of m and n. We can receive some values of m and n.

Step 4. We substitute (4) into (3) to get a polynomial $R(Q)$ and equate all the coefficients of Q^i, $(i = 0,1,2,...)$ to zero, to yield a system of algebraic equations for a_i $(i = 0,1,...,n)$ and b_j $(j = 0,1,...,m)$.

Step 5. We solve the algebraic equations obtained in Step 4 using Mathematica or Maple, to get k, λ and the coefficients of a_i $(i = 0,1,...,n)$ and b_j $(j = 0,1,...,m)$. In this way, we attain the exact solutions to (3).

The obtained solutions depend on the symmetrical hyperbolic Fibonacci functions given in [60]. The symmetrical Fibonacci sine, cosine, tangent, and cotangent functions are respectively, defined as:

$$sFs(x) = \frac{a^x - a^{-x}}{\sqrt{5}}, \qquad cFs(x) = \frac{a^x + a^{-x}}{\sqrt{5}},$$
$$\tan Fs(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}, \qquad \cot Fs(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}, \tag{10}$$

$$sFs(x) = \frac{2}{\sqrt{5}} sh(x\ln(a)), \qquad cFs(x) = \frac{2}{\sqrt{5}} ch(x\ln(a)),$$
$$\tan Fs(x) = \tanh(x\ln(a)), \qquad \cot Fs(x) = \coth(x\ln(a)). \tag{11}$$

3 Applications

In this section, we construct the exact solutions in terms of the symmetrical hyperbolic Fibonacci functions of the following three nonlinear PDEs using the generalized Kudryashov method described in Section 2:

3.1 Example 1. The diffusive predator-prey system

This equation is well-known [58, 59] and can be written in the form:

$$\begin{cases} u_t = u_{xx} - \beta u + (1+\beta)u^2 - u^3 - uv, \\ v_t = v_{xx} + kuv - mv - \delta v^3, \end{cases} \tag{12}$$

where k, δ, m and β represent positive parameters, subscripts x and t denote partial derivatives. The biological meaning of each term of (12) has been discussed in [61, 62]. Recently, Zayed et al. [59] used the modified simple equation method to solve (12). In order to investigate the dynamics of the diffusive predator-prey system, the relations between the parameters, namely $m = \beta$ and $k + 1/\sqrt{\delta} = \beta + 1$, have been defined in [61]. Under this relation, (12) can be written in the form:

$$\begin{cases} u_t = u_{xx} - \beta u + (k + 1/\sqrt{\delta})u^2 - u^3 - uv, \\ v_t = v_{xx} + kuv - \beta v - \delta v^3. \end{cases} \tag{13}$$

We proceed by considering the traveling wave transformation:

$$u(x,t) = U(\zeta), \quad v(x,t) = V(\zeta), \quad \zeta = lx - wt, \tag{14}$$

where l and w are constants with $l, w \neq 0$, to reduce the nonlinear PDEs (13) into the following nonlinear ODEs:

$$\begin{cases} l^2U'' + wU' - \beta U + (k+1/\sqrt{\delta})U^2 - U^3 - UV = 0, \\ l^2V'' + wV' + kUV - \beta V - \delta V^3 = 0. \end{cases} \tag{15}$$

In order to solve (15), let us consider the following transformation

$$V = \frac{1}{\sqrt{\delta}}U. \tag{16}$$

Substituting the transformation (16) into (15), we get

$$l^2U'' + wU' - \beta U + kU^2 - U^3 = 0. \tag{17}$$

Balancing U'' and U^3 in (17), we have the following relation:

$$n - m + 2 = 3(n - m) \Rightarrow n = m + 1. \tag{18}$$

If we choose $m = 1$ and $n = 2$, then the formal solution of (17) has the form:

$$U(\zeta) = \frac{a_0 + a_1Q + a_2Q^2}{b_0 + b_1Q}. \tag{19}$$

Consequently,

$$U'(\zeta) = Q(Q-1)\left[\frac{(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)}{(b_0 + b_1Q)^2}\right]\ln(a), \tag{20}$$

$$U''(\zeta) = Q(Q-1)(2Q-1)\cdot$$
$$\cdot\left[\frac{(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)}{(b_0 + b_1Q)^2}\right]\ln^2(a) + \frac{Q^2(Q-1)^2\ln^2(a)}{(b_0 + b_1Q)^3}\cdot \tag{21}$$
$$\cdot\left[2a_2(b_0 + b_1Q)^2 - 2b_1(a_1 + 2a_2Q)(b_0 + b_1Q) + 2b_1^2(a_0 + a_1Q + a_2Q^2)\right].$$

Substituting (19) – (21) into (17), collecting the coefficients of each power of Q^i $(i = 0,1,...,6)$ and setting each of the coefficients to zero, we obtain the following system of algebraic equations:

$$Q^6 : -a_2^3 + 2l^2a_2b_1^2\ln^2(a) = 0,$$

$$Q^5 : 6l^2a_2b_0b_1\ln^2(a) + wa_2b_1^2\ln(a) - 3l^2a_2b_1^2\ln^2(a) + ka_2^2b_1 - 3a_1a_2^2 = 0,$$

$$Q^4 : ka_2^2b_0 - wa_2b_1^2\ln(a) - \beta a_2b_1^2 + 6l^2a^2b_0^2\ln^2(a) + l^2a_2b_1^2\ln^2(a)$$
$$+ 3wa_2b_0b_1\ln(a) - 3a_0a_2^2 + 2ka_1a_2b_1 - 9l^2a_2b_0b_1\ln^2(a) - 3a_1^2a_2 = 0,$$

$$
\begin{aligned}
Q^3 \ :& -2l^2 b_1 a_0 b_0 \ln^2(a) + w a_1 b_0 b_1 \ln(a) + 2k a_0 a_2 b_1 + k a_1^2 b_1 \\
& + l^2 a_1 b_0 b_1 \ln^2(a) - a_1^3 + 3l^2 a_2 b_0 b_1 \ln^2(a) + 2l^2 a_1 b_0^2 \ln^2(a) \\
& + 2k a_1 a_2 b_0 + 2w a_2 b_0^2 \ln(a) - 3w a_2 b_0 b_1 \ln(a) - w b_1^2 a_0 \ln(a) \\
& - \beta a_1 b_1^2 - 10 l^2 a_2 b_0^2 \ln^2(a) - l^2 b_1^2 a_0 \ln^2(a) - 6 a_0 a_1 a_2 - 2\beta a_2 b_0 b_1 = 0, \\
Q^2 \ :& \ 2k a_0 a_2 b_0 + 3l^2 b_1 a_0 b_0 \ln^2(a) - 3l^2 a_1 b_0^2 \ln^2(a) + w b_1^2 a_0 \ln(a) \\
& - w b_1 a_0 b_0 \ln(a) - 3a_0^2 a_2 - \beta b_1^2 a_0 + k a_1^2 b_0 - 2\beta a_1 b_0 b_1 - l^2 a_1 b_0 b_1 \ln^2(a) \\
& - 2w a_2 b_0^2 \ln(a) + 2k a_0 a_1 b_1 - \beta a_2 b_0^2 + l^2 b_1^2 a_0 \ln^2(a) + w a_1 b_0^2 \ln(a) \\
& - w a_1 b_0 b_1 \ln(a) + 4l^2 a_2 b_0^2 \ln^2(a) - 3a_0 a_1^2 = 0, \\
Q^1 \ :& \ l^2 a_1 b_0^2 \ln^2(a) - 2\beta b_1 a_0 b_0 + w b_1 a_0 b_0 \ln(a) - l^2 b_1 a_0 b_0 \ln^2(a) \\
& + 2k a_0 a_1 b_0 - w a_1 b_0^2 \ln(a) + k a_0^2 b_1 - \beta a_1 b_0^2 - 3a_0^2 a_1 = 0, \\
Q^0 \ :& -a_0^3 - \beta a_0 b_0^2 + k a_0^2 b_0 = 0.
\end{aligned}
\tag{22}
$$

Solving the system of algebraic equations (22) by Maple or Mathematica, we obtain the following set of solutions:

<u>Set 1</u>:

$$
\begin{gathered}
l = \frac{a_2}{\sqrt{2} b_1 \ln(a)}, \ w = \frac{k a_2}{2 b_1 \ln(a)}, \ \beta = \frac{b_1^2 k^2 - a_2^2}{4 b_1^2}, \\
a_0 = \frac{b_0 (k b_1 - a_2)}{2 b_1}, a_1 = \frac{k b_1^2 - b_1 a_2 + 2 a_2 b_0}{2 b_1}, \\
a_2 = a_2, b_0 = b_0, b_1 = b_1, k = k.
\end{gathered}
\tag{23}
$$

Substituting (23) into (19), we get the following solution:

$$
U(\zeta) = \left[\frac{k}{2} - \frac{a_2 (a^{\zeta} \mp 1)}{2 b_1 (a^{\zeta} \pm 1)} \right]. \tag{24}
$$

With the help of (10) and (11), the exact solutions of (12) have the forms:

$$
\begin{aligned}
u(x,t) &= \frac{k}{2} - \frac{a_2}{2 b_1} \tan Fs \left[\frac{a_2}{4 b_1 \ln(a)} \left(\sqrt{2} x - kt \right) \right] \\
&= \frac{1}{2} \left[k - \frac{a_2}{b_1} \tanh \left(\frac{a_2}{4 b_1} \left(\sqrt{2} x - kt \right) \right) \right],
\end{aligned}
\tag{25}
$$

$$
v(x,t) = \frac{1}{2\sqrt{\delta}} \left[k - \frac{a_2}{b_1} \tanh \left(\frac{a_2}{4 b_1} \left(\sqrt{2} x - kt \right) \right) \right], \tag{26}
$$

or

$$u(x,t)=\frac{1}{2}k-\frac{a_2}{2b_1}\cot Fs\left[\frac{a_2}{4b_1\ln(a)}\left(\sqrt{2}x-kt\right)\right]$$
$$=\frac{1}{2}\left[k-\frac{a_2}{b_1}\coth\left(\frac{a_2}{4b_1}\left(\sqrt{2}x-kt\right)\right)\right], \tag{27}$$

$$v(x,t)=\frac{1}{2\sqrt{\delta}}\left[k-\frac{a_2}{b_1}\coth\left(\frac{a_2}{4b_1}\left(\sqrt{2}x-kt\right)\right)\right]. \tag{28}$$

<u>Set 2</u>:

$$w=-l(3l\ln(a)\mp\sqrt{2}k),\ \beta=-l\ln(a)(2l\ln(a)\mp\sqrt{2}k),\ a_0=\pm\sqrt{2}lb_0\ln(a),$$
$$a_1=\mp\sqrt{2}l(b_0-b_1)\ln(a),\ a_2=\mp\sqrt{2}lb_1\ln(a),\ b_0=b_0,\ b_1=b_1,\ l=l,\ k=k. \tag{29}$$

For this set, we have the exact solutions of (12) in the forms:

$$u(x,t)=\pm\frac{l\ln(a)}{\sqrt{2}}\left(1+\tanh\eta\right), \tag{30}$$

$$v(x,t)=\pm\frac{l\ln(a)}{\sqrt{2}\sqrt{\delta}}\left(1+\tanh\eta\right), \tag{31}$$

or

$$u(x,t)=\pm\frac{l\ln(a)}{\sqrt{2}}\left(1+\coth\eta\right), \tag{32}$$

$$v(x,t)=\pm\frac{l\ln(a)}{\sqrt{2}\sqrt{\delta}}\left(1+\coth\eta\right), \tag{33}$$

where

$$\eta=\frac{l\ln(a)}{2}\left[x+(3l\ln(a)\mp\sqrt{2}k)t\right].$$

<u>Set 3</u>**:**

$$l=\frac{-3b_1k}{2\sqrt{2}(b_0+2b_1)\ln(a)},\ w=\frac{3k^2b_1(4b_0-b_1)}{8(b_0+2b_1)^2\ln(a)},\ \beta=\frac{3k^2b_1(b_1+2b_0)}{4(b_0+2b_1)^2},$$
$$a_0=\frac{3kb_0b_1}{2(b_0+2b_1)},\ a_1=\frac{-3kb_1(b_0-b_1)}{2(b_0+2b_1)},\ a_2=\frac{-3kb_1^{\ 2}}{2(b_0+2b_1)},\ b_0=b_0, b_1=b_1, k=k. \tag{34}$$

For this set, we have the exact solutions of (12) in the forms:

$$u(x,t)=\frac{3kb_1}{4(b_0+2b_1)}\left(1+\tanh\eta\right), \tag{35}$$

$$v(x,t)=\frac{3kb_1}{4\sqrt{\delta}(b_0+2b_1)}\left(1+\tanh\eta\right), \tag{36}$$

or

$$u(x,t)=\frac{3kb_1}{4(b_0+2b_1)}\left(1+\coth\eta\right), \tag{37}$$

$$v(x,t)=\frac{3kb_1}{4\sqrt{\delta}(b_0+2b_1)}\left(1+\coth\eta\right), \tag{38}$$

where

$$\eta=\frac{-3kb_1}{16(b_0+2b_1)^2}\left[2\sqrt{2}(b_0+2b_1)x+k(4b_0-b_1)t\right].$$

<u>Set 4</u>:

$$\begin{gathered} l=\frac{a_2}{\sqrt{2}b_1\ln(a)},\ w=\frac{ka_2}{2b_1\ln(a)},\ \beta=\frac{b_1^2k^2-4a_2^2}{4b_1^2},\ a_0=\frac{1}{4}(2a_2-kb_1), \\ a_1=\frac{-1}{2}(2a_2-kb_1),\ b_0=\frac{-1}{2}b_1,\ a_2=a_2,\ b_1=b_1,\ k=k. \end{gathered} \tag{39}$$

For this set, the exact solutions of (12) have the forms:

$$u(x,t)=\frac{1}{2}k-\frac{a_2}{b_1}\coth\left[\frac{a_2}{2b_1}\left(\sqrt{2}x-kt\right)\right], \tag{40}$$

$$v(x,t)=\frac{1}{2b_1\sqrt{\delta}}\left[kb_1-2a_2\coth\left(\frac{a_2}{2b_1}\left(\sqrt{2}x-kt\right)\right)\right]. \tag{41}$$

<u>Set 5</u>:

$$\begin{gathered} w=-l(3l\ln(a)\mp\sqrt{2}k),\beta=-l(2l\ln(a)\mp\sqrt{2}k)\ln(a),a_0=\pm\sqrt{2}lb_0\ln(a), \\ a_1=\mp\sqrt{2}lb_0\ln(a),a_2=0,b_0=b_0,b_1=b_1,k=k,l=l. \end{gathered} \tag{42}$$

For this set, we have the exact solutions of (12) in the forms:

$$u(x,t)=\pm\frac{\sqrt{2}lb_0\ln(a)\left[1+\tanh\eta\right]}{2b_0+b_1\left[1-\tanh\eta\right]}, \tag{43}$$

$$v(x,t)=\pm\sqrt{\frac{2}{\delta}}\left(\frac{lb_0\ln(a)\left[1+\tanh\eta\right]}{2b_0+b_1\left[1-\tanh\eta\right]}\right), \tag{44}$$

or

$$u(x,t)=\pm\frac{\sqrt{2}lb_0\ln(a)\left[1+\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}, \tag{45}$$

$$v(x,t)=\pm\sqrt{\frac{2}{\delta}}\left(\frac{lb_0\ln(a)\left[1+\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}\right), \tag{46}$$

where

$$\eta = \frac{l}{2}\left[x + (3l\ln(a) \mp \sqrt{2}k)t\right]\ln(a).$$

Set 6:

$$\begin{gathered} l = \frac{a_0 b_1 - a_1 b_0}{\sqrt{2} b_0 (b_0 + b_1)\ln(a)}, w = \frac{-2(a_0 b_0 + a_0 b_1 + a_1 b_0)(a_0 b_1 - a_1 b_0)}{2 b_0^{\,2} (b_0 + b_1)^2 \ln(a)}, a_2 = 0, \\ \beta = \frac{a_0 (a_0 + a_1)}{b_0 (b_0 + b_1)},\ k = \frac{a_1 b_0 + a_0 (2b_0 + b_1)}{b_0 (b_0 + b_1)},\ a_0 = a_0, b_0 = b_0,\ a_1 = a_1,\ b_1 = b_1. \end{gathered} \tag{47}$$

For this set, we have the exact solutions of (12) in the forms:

$$u(x,t) = \frac{2a_0 + a_1\left[1 - \tanh\eta\right]}{2b_0 + b_1\left[1 - \tanh\eta\right]}, \tag{48}$$

$$v(x,t) = \frac{1}{\sqrt{\delta}}\left(\frac{2a_0 + a_1\left[1 - \tanh\eta\right]}{2b_0 + b_1\left[1 - \tanh\eta\right]}\right), \tag{49}$$

or

$$u(x,t) = \frac{2a_0 + a_1\left[1 - \coth\eta\right]}{2b_0 + b_1\left[1 - \coth\eta\right]}, \tag{50}$$

$$v(x,t) = \frac{1}{\sqrt{\delta}}\left(\frac{2a_0 + a_1\left[1 - \coth\eta\right]}{2b_0 + b_1\left[1 - \coth\eta\right]}\right), \tag{51}$$

where

$$\eta = \frac{\sqrt{2}(a_0 b_1 - a_1 b_0)}{4b_0 (b_0 + b_1)}x + \frac{(2a_0 b_0 + a_0 b_1 + a_1 b_0)(a_0 b_1 - a_1 b_0)}{4b_0^2 (b_0 + b_1)^2}t.$$

Set 7:

$$\begin{gathered} w = l(3l\ln(a) \mp \sqrt{2}k),\ \ \beta = -l(2l\ln(a) \mp \sqrt{2}k)\ln(a),\ \ a_0 = 0, \\ a_1 = \pm\sqrt{2}l(b_0 + b_1)\ln(a),\ \ a_2 = 0,\ \ b_0 = b_0,\ b_1 = b_1,\ \ k = k,\ l = l. \end{gathered} \tag{52}$$

For this set, we have the exact solutions of (12) in the forms:

$$u(x,t) = \pm\frac{\sqrt{2}l(b_0 + b_1)\ln(a)\left[1 - \tanh\eta\right]}{2b_0 + b_1\left[1 - \tanh\eta\right]}, \tag{53}$$

$$v(x,t) = \pm\sqrt{\frac{2}{\delta}}\left(\frac{l(b_0 + b_1)\ln(a)\left[1 - \tanh\eta\right]}{2b_0 + b_1\left[1 - \tanh\eta\right]}\right), \tag{54}$$

or

$$u(x,t)=\pm\frac{\sqrt{2}l(b_0+b_1)\ln(a)\left[1-\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}, \tag{55}$$

$$v(x,t)=\pm\sqrt{\frac{2}{\delta}}\left(\frac{l(b_0+b_1)\ln(a)\left[1-\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}\right), \tag{56}$$

where

$$\eta=\frac{l}{2}\left[x-(3l\ln(a)\mp\sqrt{2}k)t\right]\ln(a).$$

<u>Set 8</u>:

$$\begin{gathered} l=\frac{k\pm\sqrt{k^2-4\beta}}{2\sqrt{2}\ln(a)},\ w=\frac{1}{\ln(a)}\left[\frac{k^2}{4}\pm\frac{k\sqrt{k^2-4\beta}}{4}-\frac{3\beta}{2}\right],\ a_0=0, \\ a_1=0,\ a_2=\frac{b_1}{2}\left(k\pm\sqrt{k^2-4\beta}\right),\ b_0=0,\ b_1=b_1,\ k=k. \end{gathered} \tag{57}$$

provided that $k^2-4\beta\geq 0$.

Substituting (57) into (19), we get the following solution:

$$U(\zeta)=\pm\frac{1}{2}\left(k\pm\sqrt{k^2-4\beta}\right)\left(\frac{1}{a^{\zeta}\pm 1}\right). \tag{58}$$

With the help of (10) and (11), the exact solutions of (12) have the forms:

$$u(x,t)=\frac{1}{4}\left(k\pm\sqrt{k^2-4\beta}\right)\left[1-\tanh\eta\right], \tag{59}$$

$$v(x,t)=\frac{1}{4\sqrt{\delta}}\left(k\pm\sqrt{k^2-4\beta}\right)\left[1-\tanh\eta\right], \tag{60}$$

or

$$u(x,t)=\frac{1}{4}\left(k\pm\sqrt{k^2-4\beta}\right)\left[1-\coth\eta\right], \tag{61}$$

$$v(x,t)=\frac{1}{4\sqrt{\delta}}\left(k\pm\sqrt{k^2-4\beta}\right)\left[1-\coth\eta\right], \tag{62}$$

where

$$\eta=\frac{1}{4\sqrt{2}}\left(k\pm\sqrt{k^2-4\beta}\right)x-\frac{1}{8}\left(k^2\pm k\sqrt{k^2-4\beta}-6\beta\right)t.$$

<u>Set 9</u>:

$$l=\pm\frac{\sqrt{k^2-4\beta}}{\sqrt{2}\ln(a)},\ w=\pm\frac{k\sqrt{k^2-4\beta}}{2\ln(a)},\ a_1=\pm b_0\sqrt{k^2-4\beta}$$
$$a_0=b_0\left[\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\right],\ a_2=0,\ b_0=b_0,\ b_1=0,\ k=k, \tag{63}$$

provided that $k^2-4\beta>0$.

Substituting (63) into (19), we get the following solution:

$$U(\zeta)=\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\pm\sqrt{k^2-4\beta}\left(\frac{1}{a^{\zeta}\pm 1}\right). \tag{64}$$

With the help of (10) and (11), the exact solutions of (1) have the forms:

$$u(x,t)=\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\pm\frac{1}{2}\sqrt{k^2-4\beta}\left[1-\tanh\eta\right], \tag{65}$$

$$v(x,t)=\frac{1}{\sqrt{\delta}}\left[\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\pm\frac{1}{2}\sqrt{k^2-4\beta}\left(1-\tanh\eta\right)\right], \tag{66}$$

or

$$u(x,t)=\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\pm\frac{1}{2}\sqrt{k^2-4\beta}\left[1-\coth\eta\right], \tag{67}$$

$$v(x,t)=\frac{1}{\sqrt{\delta}}\left[\frac{\pm k\sqrt{k^2-4\beta}-(k^2-6\beta)}{k\pm 3\sqrt{k^2-4\beta}}\pm\frac{1}{2}\sqrt{k^2-4\beta}\left(1-\coth\eta\right)\right], \tag{68}$$

where

$$\eta=\pm\frac{1}{4}\sqrt{k^2-4\beta}\left[\sqrt{2}x-kt\right].$$

On comparing our results (30) – (33), with the results (26) – (29) obtained in [59], we deduce that they are equivalent in the special case with $l=1,\ a=e,\ c=w$ while our results (25) – (28), (35) – (38), (40), (41), (43) – (46), (48) – (51), (53) – (56), (59) – (62) and (65) – (68) are new, and not discussed elsewhere.

3.2 Example 2. The nonlinear Bogoyavlenskii equations

In this subsection, we apply the given method to solve the following nonlinear Bogoyavlenskii equations [59, 63 – 65]:

$$\begin{cases} 4u_t + u_{xxy} - 4u^2 u_y - 4u_x v = 0, \\ uu_y = v_x. \end{cases} \tag{69}$$

In [65], the Lax pair and a nonisospectral condition for the spectral parameter are presented. Equations (69) were again derived by Kudryashov and Pickering [66] as a member of a (2+1) Schwarzian breaking soliton hierarchy, and rational solutions of it were obtained. Equations (69) also appeared in [67] as one of the equations associated with nonisospectral scattering problems. The Painleve property of (69) is checked by Estevez et al. [68]. Recently, Zayed et al. [59] used the modified simple equation method to solve (69).

This equation can be considered as the modified version of the breaking soliton equation $4u_{xt} + 8u_x u_{xy} + 4u_y u_{xx} + u_{xxxy} = 0,$ which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the *y*-axis with a long wave along the *x*-axis [65].

It is well-known that the solutions and its dynamics of the nonlinear PDEs can make researchers deeply understand the described physical process. To this aim, we use the wave transformation

$$u(x,y,t) = U(\zeta), \quad v(x,y,t) = V(\zeta), \qquad \zeta = x + y - ct, \tag{70}$$

where c is an arbitrary constant with $c \neq 0,$ to reduce (69) to the following nonlinear system of ODEs:

$$\begin{cases} -4cU' + U''' - 4U^2 U' - 4U'V = 0, \\ \frac{1}{2}U^2 = V. \end{cases} \tag{71}$$

Substituting the second equation of (71) into the first one, and integrating the resultant equation with respect to ζ and vanishing the constant of integration, we obtain

$$U'' - 2U^3 - 4cU = 0. \tag{72}$$

By balancing U'' with U^3, we have $n=m+1$. If we choose $m=1$ and $n=2$, then (72) has the same formal solutions (19).

Substituting (19) and (21) into (72), and equating all the coefficients of Q^i $(i = 0,1,...,6)$ to zero, we obtain the following system of algebraic equations:

$$Q^6 : -2a_2^3 + 2a_2 b_1^2 \ln^2(a) = 0,$$

$$Q^5 : -6a_1 a_2^2 - 3a_2 b_1^2 \ln^2(a) + 6a_2 b_0 b_1 \ln^2(a) = 0,$$

$$Q^4 : -6a_1^2 a_2 - 4ca_2 b_1^2 + a_2 b_1^2 \ln^2(a) - 9a_2 b_0 b_1 \ln^2(a) + 6a_2 b_0^2 \ln^2(a) - 6a_0 a_2^2 = 0$$

$$Q^3 : -10a_2b_0^2\ln^2(a) - b_1^2a_0\ln^2(a) - 2a_1^3 - 12a_0a_1a_2 + 2a_1b_0^2\ln^2(a) - 4ca_1b_1^2$$
$$-2b_1a_0b_0\ln^2(a) - 8ca_2b_0b_1 + a_1b_0b_1\ln^2(a) + 3a_2b_0b_1\ln^2(a) = 0,$$

$$Q^2 : b_1^2a_0\ln^2(a) - 4ca_0b_1^2 - 3a_1b_0^2\ln^2(a) - 4ca_2b_0^2 + 3b_1a_0b_0\ln^2(a) - 6a_0^2a_2$$
$$+4a_2b_0^2\ln^2(a) - 8ca_1b_0b_1 - a_1b_0b_1\ln^2(a) - 6a_0a_1^2 = 0,$$

$$Q^1 : -8ca_0b_0b_1 - 4ca_1b_0^2 + a_1b_0^2\ln^2(a) - 6a_0^2a_1 - b_1a_0b_0\ln^2(a) = 0, \tag{73}$$

$$Q^0 : -2a_0^3 - 4ca_0b_0^2 = 0.$$

On solving the above set of algebraic equations (73) with the aid of Maple or Mathematica, we get the following cases:

Case 1:

$$a_0 = 0,\ a_1 = -\frac{1}{2}a_2,\ b_0 = 0,\ b_1 = \mp\frac{a_2}{\ln(a)},\ c = -\frac{1}{8}\ln^2(a),\ a_2 = a_2. \tag{74}$$

With the help of (10) and (11), the exact solutions of (69) are in the forms:

$$u(x,y,t) = \pm\left(\frac{1}{2}\ln(a)\right)\tanh\eta, \tag{75}$$

$$v(x,y,t) = \left(\frac{1}{8}\ln^2(a)\right)\tanh^2\eta, \tag{76}$$

or

$$u(x,y,t) = \pm\left(\frac{1}{2}\ln(a)\right)\coth\eta, \tag{77}$$

$$v(x,y,t) = \left(\frac{1}{8}\ln^2(a)\right)\coth^2\eta, \tag{78}$$

where

$$\eta = \frac{1}{2}\left[x + y + \left(\frac{1}{8}\ln^2(a)\right)t\right]\ln(a).$$

Case 2:

$$a_0 = 0, a_2 = -a_1,\ b_0 = \pm\frac{a_1}{2\ln(a)},\ b_1 = \mp\frac{a_1}{\ln(a)},\ c = \frac{1}{4}\ln^2(a),\ a_1 = a_1. \tag{79}$$

With the help of (10) and (11), the exact solutions of (69) are in the forms:

$$u(x,y,t) = \pm\left(\ln(a)\right)\operatorname{csch}\eta, \tag{80}$$

$$v(x,y,t)=\left(\frac{1}{2}\ln^2(a)\right)\operatorname{csch}^2\eta, \tag{81}$$

where

$$\eta=\left[x+y-\left(\frac{1}{4}\ln^2(a)\right)t\right]\ln(a).$$

Case 3:

$$a_0=a_0,\ a_1=\mp(b_1\ln(a)\pm 4a_0),\ b_0=\pm\frac{2a_0}{\ln(a)},\ b_1=b_1,\ c=-\frac{1}{8}\ln^2(a),\ a_2=0. \tag{82}$$

With the help of (10) and (11), the exact solutions of (69) are in the forms:

$$u(x,y,t)=\pm\left[\frac{-b_1\ln(a)+k_1\tanh\eta}{2\left(k_1-b_1\ln(a)\tanh\eta\right)}\right]\ln(a), \tag{83}$$

$$v(x,y,t)=\frac{1}{8}\ln^2(a)\left[\frac{-b_1\ln(a)+k_1\tanh\eta}{k_1-b_1\ln(a)\tanh\eta}\right]^2, \tag{84}$$

or

$$u(x,y,t)=\pm\left[\frac{-b_1\ln(a)+k_1\coth\eta}{2\left(k_1-b_1\ln(a)\coth\eta\right)}\right]\ln(a), \tag{85}$$

$$v(x,y,t)=\frac{1}{8}\ln^2(a)\left[\frac{-b_1\ln(a)+k_1\coth\eta}{k_1-b_1\ln(a)\coth\eta}\right]^2, \tag{86}$$

where

$$k_1=b_1\ln(a)\pm 4a_0,\ \ \eta=\frac{1}{2}\left[x+y+\left(\frac{1}{8}\ln^2(a)\right)t\right]\ln(a).$$

3.3 Example 3. The nonlinear telegraph equation

Here we apply the method described in Section 2 to construct new exact solutions of the nonlinear telegraph equation [58, 69]:

$$u_{tt}-u_{xx}+u_t+\alpha u+\beta u^3=0. \tag{87}$$

Equation (87) is referred to as second-order hyperbolic telegraph equation with constant coefficients which models a mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation [70]. But (87) is commonly used in the signal analysis for transmission and propagation of electrical signals [70]. Equations of this kind arise in the study of heat transfer, transmission lines,

chemical kinetics, biological population dispersal, random walks (see [70, and references therein]).

We use the wave transformation

$$u(x,t)=U(\zeta),\quad \zeta=lx-wt, \tag{88}$$

where l and w are arbitrary constants with $l,w\neq 0,$ to reduce (87) to the following nonlinear ODE:

$$(w^2-l^2)U''-wU'+\alpha U+\beta U^3=0. \tag{89}$$

By balancing U'' with U^3, we have $n=m+1$. If we choose $m=1$ and $n=2$, then (89) has the same formal solutions (19).

Substituting (19) − (21) into (89), and equating all the coefficients of Q^i $(i=0,1,...,6)$ to zero, we obtain the following system of algebraic equations:

$$\begin{aligned}
&Q^6: 2l^2a_2b_1^2\ln^2(a)-\beta a_2^3-2w^2a_2b_1^2\ln^2(a)=0,\\
&Q^5: 6l^2a_2b_0b_1\ln^2(a)-3\beta a_1a_2^2+3w^2a_2b_1^2\ln^2(a)-3l^2a_2b_1^2\ln^2(a)\\
&\quad +wa_2b_1^2\ln(a)-6w^2a_2b_0b_1\ln^2(a)=0,\\
&Q^4: -3\beta a_1^2a_2+l^2a_2b_1^2\ln^2(a)-wa_2b_1^2\ln(a)-9l^2a_2b_0b_1\ln^2(a)\\
&\quad -6w^2a_2b_0^2\ln^2(a)+9w^2a_2b_0b_1\ln^2(a)-\alpha a_2b_1^2+6l^2a_2b_0^2\ln^2(a)\\
&\quad +3wa_2b_0b_1\ln(a)-3\beta a_0a_2^2-w^2a_2b_1^2\ln^2(a)=0,\\
&Q^3: -2l^2b_1a_0b_0\ln^2(a)-3w^2a_2b_0b_1\ln^2(a)+2w^2b_1a_0b_0\ln^2(a)+2wa_2b_0^2\ln(a)\\
&\quad -10l^2a_2b_0^2\ln^2(a)+2l^2a_1b_0^2\ln^2(a)+10w^2a_2b_0^2\ln^2(a)-l^2b_1^2a_0\ln^2(a)\\
&\quad +l^2a_1b_0b_1\ln^2(a)-w^2a_1b_0b_1\ln^2(a)-\beta a_1^3-\alpha a_1b_1^2+3l^2a_2b_0b_1\ln^2(a)\\
&\quad -3wa_2b_0b_1\ln(a)-wb_1^2a_0\ln(a)-2w^2a_1b_0^2\ln^2(a)+wa_1b_0b_1\ln(a)\\
&\quad -6\beta a_0a_1a_2+w^2b_1^2a_0\ln^2(a)-2\alpha a_2b_0b_1=0,\\
&Q^2: 4l^2a_2b_0^2\ln^2(a)-l^2a_1b_0b_1\ln^2(a)-\alpha b_1^2a_0-w^2b_1^2a_0\ln^2(a)-\alpha a_2b_0^2\\
&\quad -wb_1a_0b_0\ln(a)-3l^2a_1b_0^2\ln^2(a)-wa_1b_0b_1\ln(a)-3\beta a_0a_1^2\\
&\quad -2wa_2b_0^2\ln(a)+3w^2a_1b_0^2\ln^2(a)+wb_1^2a_0\ln(a)+wa_1b_0^2\ln(a)\\
&\quad -2\alpha a_1b_0b_1-3w^2b_1a_0b_0\ln^2(a)+3l^2b_1a_0b_0\ln^2(a)\\
&\quad -4w^2a_2b_0^2\ln^2(a)+w^2a_1b_0b_1\ln^2(a)-3\beta a_0^2a_2+l^2b_1^2a_0\ln^2(a)=0,\\
&Q^1: -wa_1b_0^2\ln(a)+w^2b_1a_0b_0\ln^2(a)-w^2a_1b_0^2\ln^2(a)-3\beta a_0^2a_1-2\alpha b_1a_0b_0\\
&\quad +wb_1a_0b_0\ln(a)-l^2b_1a_0b_0\ln^2(a)-\alpha a_1b_0^2+l^2a_1b_0^2\ln^2(a)=0,
\end{aligned} \tag{90}$$

$Q^0 : \beta a_0^3 + \alpha a_0 b_0^2 = 0.$

Solving the above algebraic equations (90) with the aid of Maple or Mathematica, the following cases of solutions are obtained:

Case 1:

$$a_0 = \mp\sqrt{-\frac{\alpha}{\beta}}b_0,\ a_1 = \pm\sqrt{-\frac{\alpha}{\beta}}(b_0 - b_1),\ a_2 = \pm\sqrt{-\frac{\alpha}{\beta}}b_1,$$
$$b_0 = b_0,\ b_1 = b_1,\ l = \pm\frac{\sqrt{9\alpha^2 - 2\alpha}}{2\ln(a)},\ w = \frac{3\alpha}{2\ln(a)}, \tag{91}$$

provided that $9\alpha^2 - 2\alpha > 0, \alpha\beta < 0.$

For this case, we have the exact solutions:

$$u(x,t) = \mp\frac{1}{2}\sqrt{\frac{-\alpha}{\beta}}\left[1 + \tanh\eta\right], \tag{92}$$

or

$$u(x,t) = \mp\frac{1}{2}\sqrt{\frac{-\alpha}{\beta}}\left[1 + \coth\eta\right], \tag{93}$$

where

$$\eta = \pm\frac{1}{4}\sqrt{9\alpha^2 - 2\alpha}x - \frac{3}{4}\alpha t.$$

Case 2:

$$a_0 = \pm\sqrt{\frac{-\alpha}{\beta}}b_0,\ a_1 = \mp\sqrt{\frac{-\alpha}{\beta}}b_0,\ a_2 = 0,\ b_0 = b_0,$$
$$b_1 = b_1,\ l = \pm\frac{\sqrt{9\alpha^2 - 2\alpha}}{2\ln(a)},\ w = \frac{3\alpha}{2\ln(a)}, \tag{94}$$

provided that $9\alpha^2 - 2\alpha > 0, \alpha\beta < 0.$

For this case, we have the exact solutions:

$$u(x,t) = \pm\frac{b_0\sqrt{\frac{-\alpha}{\beta}}\left[1 + \tanh\eta\right]}{2b_0 + b_1\left[1 - \tanh\eta\right]}, \tag{95}$$

or

$$u(x,t)=\pm\frac{b_0\sqrt{\frac{-\alpha}{\beta}}\left[1+\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}, \tag{96}$$

where

$$\eta=\pm\frac{1}{4}\sqrt{9\alpha^2-2\alpha}\,x-\frac{3}{4}\alpha t.$$

<u>Case 3:</u>

$$\begin{gathered} a_0=0,\ a_1=\pm\sqrt{\frac{-\alpha}{\beta}}(b_0+b_1),\ a_2=0,\ b_0=b_0, \\ b_1=b_1,\ l=\pm\frac{\sqrt{9\alpha^2-2\alpha}}{2\ln(a)},\ w=\frac{-3\alpha}{2\ln(a)}, \end{gathered} \tag{97}$$

provided that $9\alpha^2-2\alpha>0,\alpha\beta<0.$

For this case, we have the exact solutions:

$$u(x,t)=\pm\frac{\sqrt{\frac{-\alpha}{\beta}}(b_0+b_1)\left[1-\tanh\eta\right]}{2b_0+b_1\left[1-\tanh\eta\right]}, \tag{98}$$

or

$$u(x,t)=\pm\frac{\sqrt{\frac{-\alpha}{\beta}}(b_0+b_1)\left[1-\coth\eta\right]}{2b_0+b_1\left[1-\coth\eta\right]}, \tag{99}$$

where

$$\eta=\pm\frac{1}{4}\sqrt{9\alpha^2-2\alpha}\,x+\frac{3}{4}\alpha t.$$

<u>Case 4:</u>

$$\begin{gathered} a_0=0,\ a_1=\pm\frac{\alpha b_1}{\beta\sqrt{\frac{-\alpha}{\beta}}},\ a_2=\pm\sqrt{\frac{-\alpha}{\beta}}b_1,\ b_0=0, \\ b_1=b_1,\ l=\pm\frac{\sqrt{9\alpha^2-2\alpha}}{2\ln(a)},\ w=\frac{3\alpha}{2\ln(a)}, \end{gathered} \tag{100}$$

provided that $9\alpha^2-2\alpha>0,\alpha\beta<0.$

Substituting (100) into (19), we get the following solution:

$$U(\zeta)=\pm\frac{\alpha}{\beta\sqrt{\frac{-\alpha}{\beta}}}\pm\sqrt{\frac{-\alpha}{\beta}}\left(\frac{1}{a^{\zeta}\pm 1}\right). \tag{101}$$

Consequently, we have the exact solutions:

$$u(x,t)=\pm\frac{\alpha}{\beta\sqrt{\frac{-\alpha}{\beta}}}\pm\frac{1}{2}\sqrt{\frac{-\alpha}{\beta}}\left[1-\tanh\eta\right], \tag{102}$$

or

$$u(x,t)=\pm\frac{\alpha}{\beta\sqrt{\frac{-\alpha}{\beta}}}\pm\frac{1}{2}\sqrt{\frac{-\alpha}{\beta}}\left[1-\coth\eta\right], \tag{103}$$

where

$$\eta=\pm\frac{1}{4}\sqrt{9\alpha^2-2\alpha}x-\frac{3}{4}\alpha t.$$

On comparing our result (101), with the result (31) obtained in [58], we deduce that they are equivalent in the special case with $a=e$, while our results (92), (93), (95), (96), (98), (99), (102) and (103) are new, and not discussed elsewhere.

4 Physical Explanations for Some of Our Solutions

In this section, we will illustrate the application of the results established above. Exact solutions of the results describe different nonlinear waves. For the established exact kink and anti-kink solutions with symmetrical hyperbolic Fibonacci functions are special kinds of solitary waves. Kink and anti-kink solutions have a remarkable property that keeps its identity upon interacting with other. Kink and anti-kink solutions have particle-like structures, for example, magnetic monopoles, and extended structures, like, domain walls and cosmic strings, that have implications for the cosmology of the early universe.

Let us now examine Figs. 1 – 3 as it illustrates some of our results obtained in this article. To this end, we select some special values of the parameters obtained, for example, in some of the kink and anti-kink solutions (30) to (33) of the diffusive predator-prey system whit $-10<x$, $t<10$, solutions (80) and (81) of the nonlinear Bogoyavlenskii equations with $-10<x$, $t<10$, solutions (95) and (96) of the nonlinear telegraph equation with $-10<x$, $t<10$, respectively.

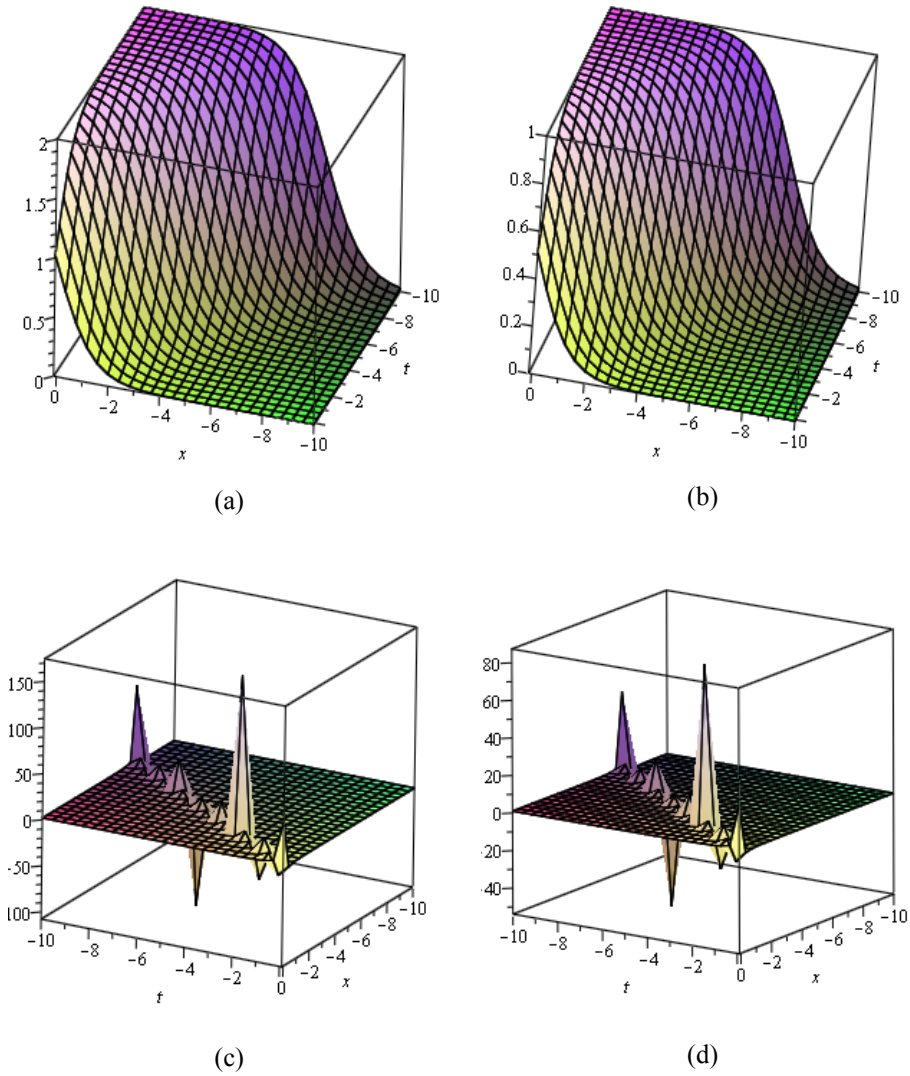
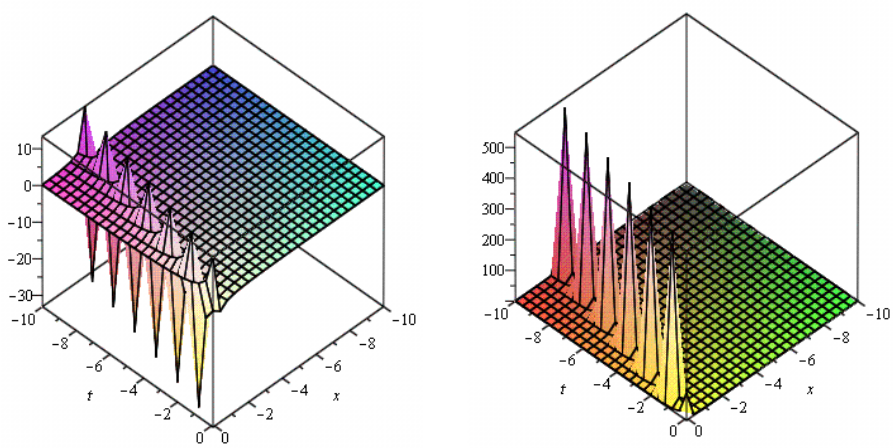
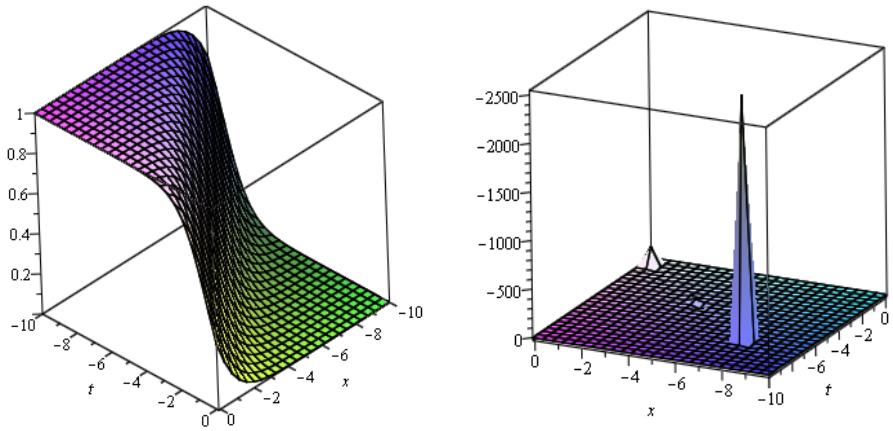


Fig. 1 – *Symmetrical Fibonacci hyperbolic function solutions of the diffusive predator-prey system when* $l=\sqrt{2}, a=e, \delta=4, k=7/2, \beta=3.$
(a) *Plot kink solution (30);* (b) *Plot kink solution (31).*
(c) *Plot anti-kink solution (32);* (d) *Plot anti-kink solution (33).*



(a) (b)

Fig. 2 – *Symmetrical Fibonacci hyperbolic function solutions of the nonlinear Bogoyavlenskii equations when* $a = e, y = 0$.
(a) *Plot solution (80);* (b) *Plot solution (81).*



(a) (b)

Fig. 3 – *Symmetrical Fibonacci hyperbolic function solutions of the nonlinear telegraph equation when* $a_0 = b_1 = \alpha = 1, \beta = -1$.
(a) *Plot solution (95);* (b) *Plot solution (96).*

5 Conclusion

In this paper, we have shown that the symmetrical hyperbolic Fibonacci function solutions can be obtained by the general Exp_a - function by using generalized Kudryashov method.We have extended successfully the generalized Kudryashov method to solve three nonlinear partial differential equations. As applications, abundant we have obtained many new symmetrical hyperbolic Fibonacci function solutions for the diffusive predator-prey system, the nonlinear Bogoyavlenskii equations, and the nonlinear telegraph equation. As one can see, that the generalized Kudryashov method is powerful, effective and convenient for solving nonlinear PDEs, the physical explanation of some solutions of these equations have been presented in Section 4. Finally, the generalized Kudryashov method provides a powerful mathematical tool to obtain more general exact analytical solutions of many nonlinear PDEs in mathematical physics. Finally, our solutions in this article have been checked using the Maple by putting them back into the original equations.

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