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Proximity Effect in a Thin Two-layer Tubular Conductor Caused by a Parallel Filament*

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Abstract: In this paper, we determine in a closed form the current distribution in a thin two-layer tubular conductor with a sinusoidal current in the presence of a filament with an equal but opposite current. The method of treating the problem consists in solving an integral equation for the current density in one layer which enables to find the current density in the other layer by imposing equality of the tangential components of the electric fields at the layer interface. An important factor, the AC to DC resistance ratio for the conductor is also found in a closed form. Some numerical examples, for current distribution and resistance ratio, versus frequency or geometry parameters, are presented.

Keywords: Skin effect, Proximity effect, Current distribution, Integral equation, AC to DC resistance ratio.

1 Introduction

Finding the current distribution in a system of parallel conductors with time-varying currents (the combined skin and proximity effects) is mathematically a very difficult problem. There are very few cases where a solution can be found in a closed form [1 – 6], and generally, various numerical methods have to be used. Most commonly used methods for analyzing skin and proximity effect problems include use of: Maxwell's equations via the magnetic vector potential, integral equations, BIE (Boundary Integral Equations), model functions, Bubnov-Galerkin method, etc.

In this paper we use integral equation method to find in a closed form the current distribution in a thin two-layer tubular conductor in the presence of a filament. A remarkable feature of this method is that solving the relevant integral equation (or a system of these equations) does not require usage of any boundary conditions, if the conductor is homogenous.

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2 Method of Solution

Fig. 1 shows the geometry of the problem, if the filament is outside. A thin tubular conductor of mean radius *a* with a sinusoidal current $i = \sqrt{2}\, I \cos\omega t$ has two layers of thicknesses *d* (*d*<<*a*), and conductivities σ_1 and σ_2 respectively. A filament with an equal and opposite current is spaced at a distance *D* apart ($D > a$). The objective is to find current distributions in the layers. Since *d* is small, we can neglect the radial dependence, so that the current densities depend only on the polar angle θ, meaning that only the proximity effect is present. As the current densities should be even functions of θ, we may seek the current densities in the form of Fourier series with the cosine harmonics. Thus, in layer 1

$$J_1(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta \tag{1}$$

and the current density in layer 2 is determined from the boundary condition for the tangential electrical fields at the layer interface, from which it follows that

$$J_2(\theta) = \frac{\sigma_2}{\sigma_1} J_1(\theta) = \chi^2 \left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta \right), \tag{2}$$

where

$$\chi^2 = \frac{\sigma_2}{\sigma_1}.$$

The constant term A_0 can be found from the known current in the conductor

$$I = \int_s J \,\mathrm{d}S = \int_0^{2\pi} J_1(\theta) ad \,\mathrm{d}\theta + \int_0^{2\pi} J_2(\theta) ad \,\mathrm{d}\theta = A_0 \left(1 + \chi^2\right) 2\pi ad \ ,$$

whence

$$A_0 = \frac{I}{2\pi ad\left(1 + \chi^2\right)}. \tag{3}$$

Note that the infinitive summations from (1) – (2) do not contribute to the integration since

$$\int_0^{2\pi} \cos n\theta \,\mathrm{d}\theta = 0, \quad n \geq 1.$$

The integral equation for current density in layer 1 is [2 – 3]:

$$\begin{aligned} J_1(\theta) = \frac{k_1^2}{4\pi} \Bigg[& \int_{S_1} J_1(\theta') \ln \frac{|\vec{r} - \vec{r}'|^2}{D^2} \mathrm{d}s + \int_{S_2} J_2(\theta') \ln \frac{|\vec{r} - \vec{r}'|^2}{D^2} \mathrm{d}s \\ & - I \ln \frac{r^2 + D^2 - 2rD\cos\theta}{D^2} \Bigg] + K_1, \quad \vec{r} = (r, \theta) \in S_1, \end{aligned} \tag{4}$$

where

$$k_1^2 = \mathrm{j}\omega\mu_0\sigma_1$$

and K_1 is an unknown constant.

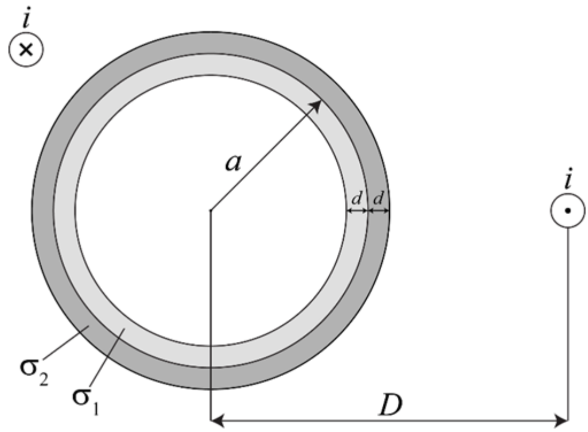

Fig. 1 – *Thin two-layer conductor and fiament spaced outside, with equal and opposite currents.*

On the right-hand side of (4), we may use the approximations

$$r \approx r' \approx a \tag{5}$$

and, by using (1), (2) and (5), (4) becomes

$$\begin{aligned} &A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta = \\ &= \frac{k_1^2}{4\pi}\left\{ ad\left(1+\chi^2\right)\left[A_0 I_0(\theta) + \sum_{n=1}^{\infty} A_n I_n(\theta) \right] + 2I\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n \frac{\cos n\theta}{n} \right\} + K_1, \end{aligned} \tag{6}$$

where

$$I_n(\theta) = \int_0^{2\pi} \cos n\theta \ln\left(\frac{4a^2}{D^2} \sin^2 \frac{\theta-\theta'}{2} \right) \mathrm{d}\theta', \quad n = 0,1,2,3,\dots, \tag{7}$$

and we also used the expansion (see Appendix 1)

$$\ln\frac{r^2 + D^2 - 2rD\cos\theta}{D^2} \approx \ln\frac{a^2 + D^2 - 2aD\cos\theta}{D^2} = -2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n \frac{\cos n\theta}{n}. \tag{8}$$

Integral (7) is evaluated in Apendix 2

$$I_n(\theta)=\begin{cases} 4\pi\ln\dfrac{a}{D}=\text{const}, & n=0 \\ -2\pi\dfrac{\cos n\theta}{n}, & n\geq 1. \end{cases} \tag{9}$$

If we replace (9) into (6) and equate the coefficients with $\cos n\theta$ on both sides we obtain

$$A_n=\frac{k_1^2}{4\pi}ad\left(1+\chi^2\right)adA_n\left(-\frac{2\pi}{n}\right)+\frac{k_1^2}{4\pi}2I\left(\frac{a}{D}\right)^n\frac{1}{n},$$

from which we find

$$A_n=\frac{I}{\pi ad\left(1+\chi^2\right)}\left(\frac{a}{D}\right)^n\frac{j\Lambda}{n+j\Lambda},\quad n=1,2,3,\ldots, \tag{10}$$

where

$$\Lambda=\pi f\mu_0\left(\sigma_1+\sigma_2\right)ad=f\cdot\text{const}.$$

The current density in layer 1, given by (1), is now determined by (3) and (10)

$$J_1(\theta)=\frac{I}{2\pi ad\left(1+\chi^2\right)}\left(1+2j\Lambda\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n\frac{\cos n\theta}{n+j\Lambda}\right) \tag{11}$$

and the current density in layer 2 is then determined from (2)

$$J_2(\theta)=\frac{\chi^2 I}{2\pi ad\left(1+\chi^2\right)}\left(1+2j\Lambda\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n\frac{\cos n\theta}{n+j\Lambda}\right). \tag{12}$$

For $\sigma_1=\sigma_2=\sigma(\chi=1)$, (11) – (12) reduce to the formula derived in [2 – 3] for a single-layer tubular conductor.

Equating the constant terms in (6) gives

$$A_0=\frac{k_1^2}{4\pi}ad\left(1+\chi^2\right)A_0I_0+K_1$$

from which the unknown constant K_1 can be found, but this constant is of no importance.

3 AC to DC Resitance Ratio of the Conductor

The power loss per unit length in layer 1 is

$$P'_{J_1} = \frac{1}{\sigma_1}\left[\int_{S_1} J_1(\theta)^2 \,\mathrm{d}S\right] = \frac{ad}{\sigma_1}\left[\int_0^{2\pi} J_1(\theta) J_1^*(\theta)\,\mathrm{d}\theta\right] =$$

$$= \frac{ad}{\sigma_1}\int_0^{2\pi}\left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta\right)\left(A_0 + \sum_{n=1}^{\infty} A_n^*(\theta)\cos n\theta\right)\mathrm{d}\theta$$

$$= \frac{ad}{\sigma_1}\left(A_0^2 2\pi + \pi\sum_{n=1}^{\infty}|A_n|^2\right) = \frac{I^2}{\sigma_1 2\pi ad\left(1+\chi^2\right)^2}\left(1 + 2\Lambda^2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\Lambda^2}\right).$$

Similarly,

$$P'_{J_2} = \frac{I^2\chi^4}{\sigma_2 2\pi ad\left(1+\chi^2\right)^2}\left(1 + 2\Lambda^2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\Lambda^2}\right).$$

The total power loss per unit length of the conductor is then

$$P'_J = P'_{J_1} + P'_{J_2} = \frac{I^2}{2\pi ad\left(\sigma_1+\sigma_2\right)}\left(1 + 2\Lambda^2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\Lambda^2}\right)$$

and the AC resistance is

$$R'_{ac} = \frac{P'_J}{I^2} = \frac{1}{2\pi ad\left(\sigma_1+\sigma_2\right)}\left(1 + 2\Lambda^2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\Lambda^2}\right). \tag{13}$$

Since the DC resistance of the conductor is

$$R'_{dc} = \frac{1}{2\pi ad\left(\sigma_1+\sigma_2\right)} \tag{14}$$

we find from (13) and (14) the AC to DC resistance ratio of the conductor

$$\frac{R'_{ac}}{R'_{dc}} = 1 + 2\Lambda^2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\Lambda^2}. \tag{15}$$

Note that this formula has the same form as the corresponding one in the case of a single-layer conductor [2 – 3].

In the case where the filament is inside the conductor (Fig. 2) we apply the same procedure, and it turns out that in the final expressions for the current densities (11) – (12) and the resistance ratio (15) a/D should be replaced by D/a.

In particular,

$$\frac{R'_{ac}}{R'_{dc}} = 1 + 2\Lambda^2 \sum_{n=1}^{\infty} \left(\frac{D}{a}\right)^{2n} \frac{1}{n^2 + \Lambda^2}. \tag{16}$$

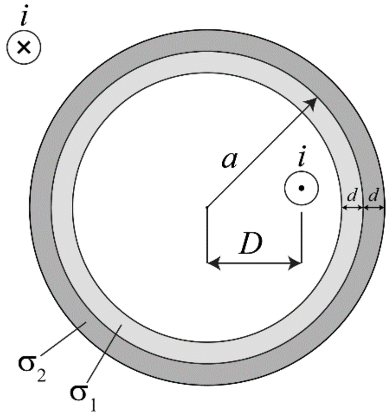

Fig. 2 *–Thin two-layer conductor and filament spaced inside, with equal and opposite currents.*

4 Numerical Results

From (15) we have calculated the AC to DC resistance ratio versus Λ (which is proportional to frequency) for different values of ratio *a*/*D*. The corresponding plots are shown in Fig. 3, for a = 3cm, $d_1 = d_2$ = 1mm, $\sigma_1 = 57 \cdot 10^6$ S/m (copper) and $\sigma_2 = 37 \cdot 10^6$ S/m (aluminium).

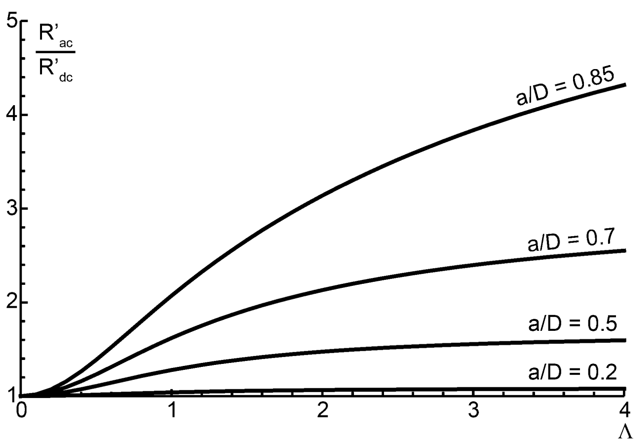

Fig. 3 – *AC to DC resistance ratio versus Λ. Ratio a/D is taken as a parameter (filament outside).*

Fig. 4 shows the normalized current distribution $\left| J_1(\theta) / \left(I / (2\pi a d) \right) \right|$ versus the polar angle θ obtained from (11), for a = 3cm, $d_1 = d_2$ = 1 mm, $\sigma_1 = 57 \cdot 10^6$ S/m (copper), $\sigma_2 = 37 \cdot 10^6$ S/m (aluminium), $\Lambda = 0.6$ and parameter a/D is varied.

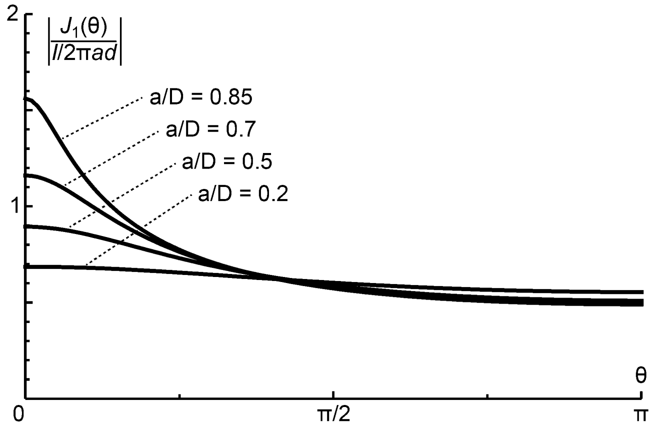

Fig. 4 – *Normalized current distribution in layer 1 of the tubular conductor for* $\Lambda = 0.6$. *Ratio a/D is taken as a parameter (filament outside).*

If Λ is taken as a parameter, the corresponding plots of the normalized current density are shown in Fig. 5, for a = 3 cm, $d_1 = d_2$ = 1 mm, $\sigma_1 = 57 \cdot 10^6$ S/m (copper), $\sigma_2 = 37 \cdot 10^6$ S/m (aluminium), $a/D = 0.85$ and for different values for Λ.

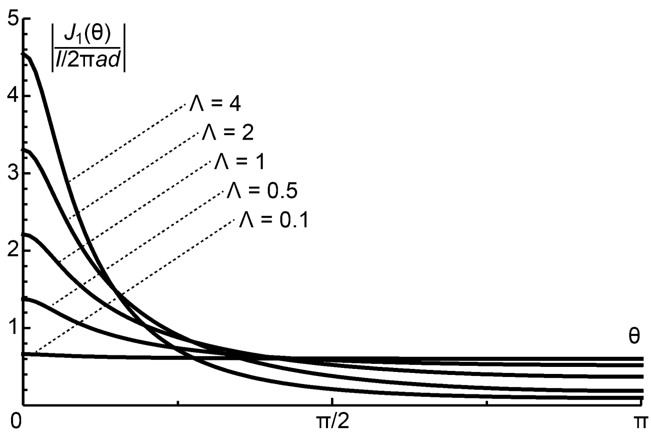

Fig. 5 – *Normalized current distribution in layer 1 of the tubular conductor for a/D* = 0.85. *Λ is taken as a parameter (filament outside).*

Finally, based on (16), Fig. 6 shows the AC to DC resistance ratio versus Λ, when the filament is inside the conductor and ratio *D*/*a* is taken as a parameter.

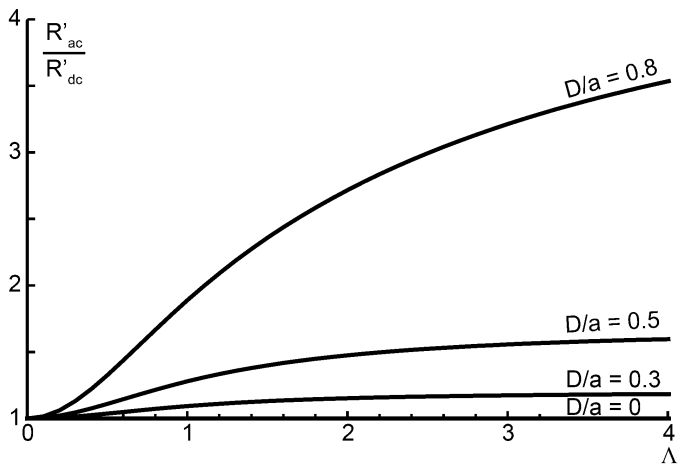

Fig. 6 – *AC to DC resistance ratio versus Λ.*
Ratio D/a is taken as a parameter (filament inside).

5 Conclusion

We presented in this paper a rigorous analysis of the proximity effect of the current filament on the current distribution in a thin two-layer tubular conductor. This distribution is found in a closed form by explicitly solving an integral equation and using a relevant boundary condition at the layer interface. An important factor, the AC to DC resistance ratio is also found in a closed form, and some numerical results are given to illustrate the developed theory. If both layers have the same conductivities (homogeneous conductor), the derived formulas coincide with those for a single-layer conductor of double thickness considered earlier

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Appendix 1

To prove (8) we start from

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| \le 1, \quad z \ne 1$$

and put $z = \frac{a}{D} e^{j\theta}$, $\left(\frac{a}{D} < 1\right)$. Then

$$\ln\left(1 - \frac{a}{D} e^{j\theta}\right) = -\sum_{n=1}^{\infty} \left(\frac{a}{D}\right)^n \frac{e^{j\theta}}{n}. \tag{A1.1}$$

Now, (8) is obtained by equating the real part in (A1.1) taking into account that

$$\operatorname{Re} \ln z = \ln|z|.$$

Appendix 2

In this Appendix we evaluate integral $I_n(\theta)$ given by (7). First, let $n = 0$. Then

$$\begin{aligned} I_0(\theta) &= \int_0^{2\pi} \ln\left(\frac{4a^2}{D^2} \sin^2 \frac{\theta-\theta'}{2}\right) d\theta' = 4\pi \ln\frac{a}{D} + 2\int_0^{2\pi} \ln\left(2\sin\frac{\theta-\theta'}{2}\right) d\theta' = \\ &= 4\pi \ln\frac{a}{D} - 2\sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\cos m(\theta-\theta')}{m} d\theta'. \end{aligned} \tag{A2.1}$$

where we used the well known Furier series

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m} = -\ln\left(2\sin\frac{x}{2}\right), \quad 0 < x < 2\pi,$$

which follows from (8) if we put $\theta = x$ and $\frac{a}{D} = 1$.

The integral on the right-hand side of (A2.1) is

$$\int_0^{2\pi} \frac{\cos m(\theta-\theta')}{m} \,\mathrm{d}\,\theta' = \frac{\cos m\theta}{m} \int_0^{2\pi} \cos m\theta' \,\mathrm{d}\,\theta' + \frac{\sin m\theta}{m} \int_0^{2\pi} \sin m\theta' \,\mathrm{d}\,\theta' = 0 \,,$$

Hence,

$$I_0(\theta) = 4\pi \ln \frac{a}{D} = \text{const}.$$

For $n \geq 1$

$$\begin{aligned} I_n(\theta) &= \int_0^{2\pi} \cos n\theta' \ln\left(\frac{4a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)\mathrm{d}\,\theta' = \\ &= \ln\frac{a^2}{D^2}\int_0^{2\pi}\cos n\theta' \,\mathrm{d}\,\theta' - 2\int_0^{2\pi}\cos n\theta' \sum_{n=1}^{\infty}\frac{\cos m(\theta-\theta')}{m}\,\mathrm{d}\,\theta'. \end{aligned} \qquad \text{(A2.2)}$$

The first integral on the right-hand side of (A2.2) vanishes (since $n \geq 1$), therefore

$$I_n(\theta) = -2\left(\sum_{n=1}^{\infty}\frac{\cos m\theta}{m}\int_0^{2\pi}\cos n\theta' \cos m\theta' \,\mathrm{d}\,\theta' + \sum_{n=1}^{\infty}\frac{\sin m\theta}{m}\int_0^{2\pi}\cos n\theta' \sin m\theta' \,\mathrm{d}\,\theta'\right),$$

$$I_n(\theta) = -2\pi\frac{\cos n\theta}{n}\,.$$

Since

$$\int_0^{2\pi} \cos n\theta' \sin m\theta' \,\mathrm{d}\,\theta' = 0 \text{, for each } m, n$$

and

$$\int_0^{2\pi} \cos m\theta' \cos m\theta' \,\mathrm{d}\,\theta' = \begin{cases} 0, & m \neq n, \\ \pi, & m = n. \end{cases}$$

Finally, from (A2.3)– (A2.5)

$$I_n(\theta) = -2\pi\frac{\cos n\theta}{n}, \quad n \geq 1$$